Exactly solvable Richardson-Gaudin models in nuclear structure

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In collaboration with people in audience:

S. Pittel, P. Schuck, P. Van Isacker.

And many others

Richardson’s Exact Solution

ume 3, number 6 PHYSICS LETTERS 1 February 1963

A RESTRICTED CLASS OF EXACT EIGENSTATES OF THE PAIRING-FORCE HAMILTONIAN *

R. W. RICHARDSON
H. M. Randall Laboratory of Physics,
University of Michigan, Ann Arbor, Michigan

Received 23 November 1962

Exact Solution of the BCS Model

$$H_P = \sum_k \varepsilon_k n_k + g \sum_{k,k'} c^\dagger_{k\uparrow} c^\dagger_{-k\downarrow} c_{-k'\downarrow} c_{k'\uparrow}$$

Eigenvalue equation:

$$H_P \left|\Psi\right\rangle = E \left|\Psi\right\rangle$$

Ansatz for the eigenstates (generalized Cooper ansatz)

$$\left|\Psi\right\rangle = \prod_{\alpha=1}^{M} \Gamma_\alpha^\dagger \left|0\right\rangle, \quad \Gamma_\alpha^\dagger = \sum_k \frac{1}{2\varepsilon_k - E_\alpha} c_{k\uparrow}^+ c_{-k\downarrow}^+$$

Richardson equations

$$1+g\sum_{k=0}\frac{1}{2\varepsilon_k-E_\alpha}+2g\sum_{\beta(\neq\alpha)=1}^{M}\frac{1}{E_\alpha-E_\beta}=0,\qquad E=\sum_{\alpha=1}^{M}E_\alpha$$

Properties:

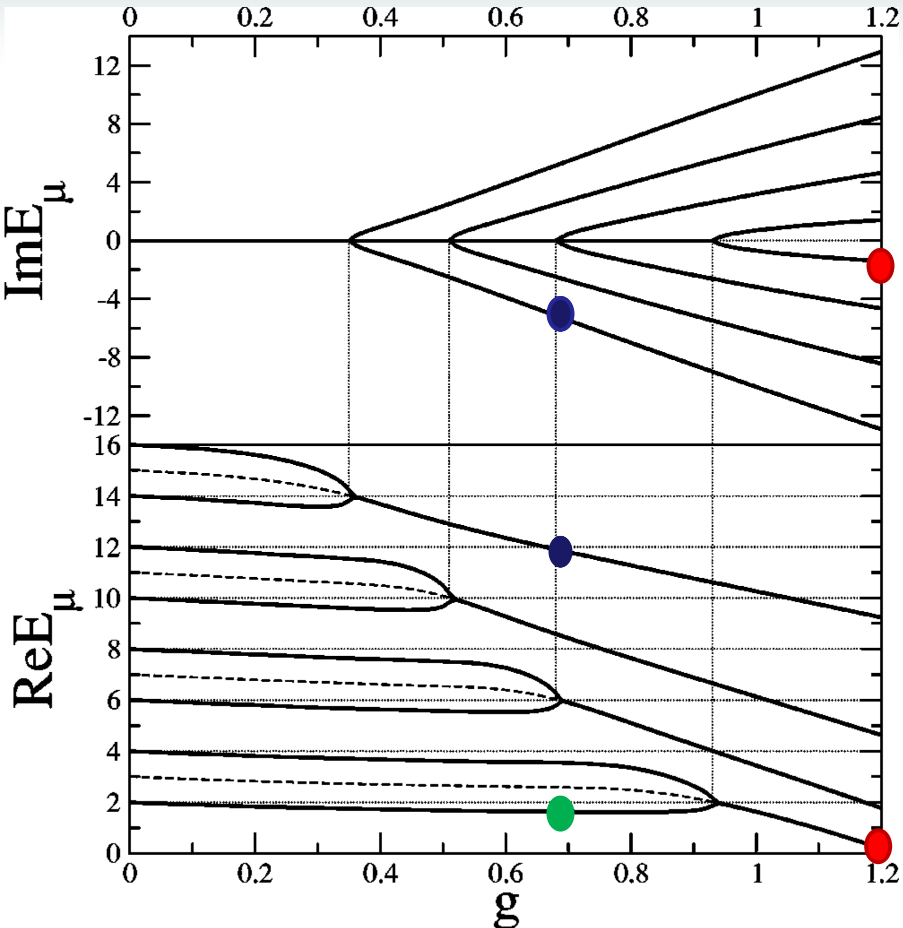
This is a set of M nonlinear coupled equations with M unknowns (E_α).

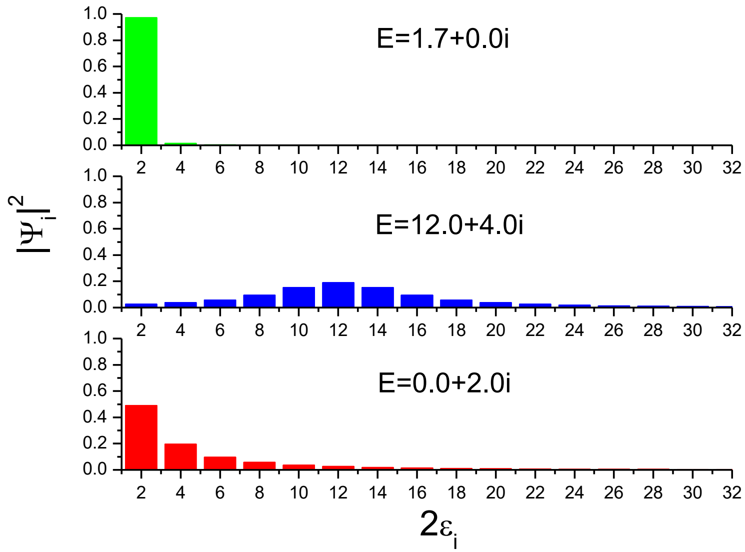
The pair energies are either real or complex conjugated pairs.

There are as many independent solutions as states in the Hilbert space. The solutions can be classified in the weak coupling limit ($g\rightarrow 0$).

Exact solvability reduces an exponential complex problem to an algebraic problem.

$$\Gamma_\alpha^\dagger = \sum_{k=1}^{L} \frac{1}{2\varepsilon_k - E_\alpha} c_{k\uparrow}^{+} c_{-k\downarrow}^{+}$$





Evolution of the real and imaginary part of the pair energies with *g*. *L=16, M=8.*

R. W. Richardson, Phys. Rev. 141 (1966) 949. Solved numerical systems up to L=32, dim=10^8

The SU(2) Algebra

Rank 1 and 1 quantum degree of freedom

$$\left[S^z, S^+\right] = S^+, \left[S^z, S^-\right] = -S^-, \quad \left[S^+, S^-\right] = 2S^z$$

The pair realizations is:

$$S_j^z = \frac{1}{2} \sum_m a_{jm}^+ a_{jm} \pm \frac{\Omega_j}{4}, \quad \mathrm{S}_j^+ = \frac{1}{2} \sum_m a_{jm}^+ a^+_{j\bar{m}}$$

Other realizations like, two level atoms, spin, finite center of mass pairs, Holstein-Primakoff or Schwinger, give rise to different physical Hamiltonians

Richardson-Gaudin Models: Construction of the Integrals of Motion

J. D., C. Esebbag and P. Schuck, Phys. Rev. Lett. 87, 066403 (2001).

•The most general combination of linear and quadratic generators, with the restriction of being hermitian and number conserving, is

$$R_i = S_i^z + 2g \sum_{j(\neq i)} \left[\frac{X_{ij}}{2} \left(S_i^+ S_j^- + S_i^- S_j^+ \right) + Y_{ij} S_i^z S_j^z \right]$$

•The integrability condition $\left[R_i , R_j \right] = 0$ leads to

$$Y_{ij} X_{jk} + X_{jk} Y_{ki} + X_{ki} X_{ij} = 0$$

•These are the same conditions encountered by Gaudin (J. de Phys. 37 (1976) 1087) in a spin model known as the Gaudin magnet.

Gaudin (1976) found three solutions

XXX (Rational)

$$X_{ij} = Y_{ij} = \frac{1}{\eta_i - \eta_j}$$

XXZ (Hyperbolic ≡ Trigonometric)

$$X_{ij} = \frac{1}{Sinh\left(x_i - x_j\right)} = 2\frac{\sqrt{\eta_i \eta_j}}{\eta_i - \eta_j}, \quad Z_{ij} = Coth\left(x_i - x_j\right) = \frac{\eta_i + \eta_j}{\eta_i - \eta_j}$$

Exact solution

$$R_i \left|\Psi\right\rangle = r_i \left|\Psi\right\rangle$$

Eigenstates of the Rational and Hyperbolic Models

Richardson ansatz

$$\left|\Psi_{XXX}\right\rangle = \prod_{\alpha}\left(\sum_i \frac{1}{\eta_i - E_\alpha} S_i^+\right)\left|0\right\rangle, \quad \left|\Psi_{XXZ}\right\rangle = \prod_{\alpha}\left(\sum_i \frac{\sqrt{\eta_i}}{\eta_i - E_\alpha} S_i^+\right)\left|0\right\rangle$$

Any function of the *R* operators defines a valid integrable Hamiltonian. The Hamiltonian is diagonal in the basis of common eigenstates of the *R* operators.

•Within the pair representation two body Hamiltonians can be obtain by a linear combination of *R* operators:

$$H = \sum_l \varepsilon_l \; R_l\left(\eta, g\right)$$

•The parameters *g*, η´*s* and ε´*s* are arbitrary. There are 2 *L*+1 free parameters to define an integrable Hamiltonian in each of the families. (*L* number of single particle levels)

• The constant PM or reduced BCS Hamiltonian solved by Richardson can be obtained by from the *XXX* family by choosing $\eta = \varepsilon$.

$$H_{BCS} = \sum_i 2\varepsilon_i S_i^z + g\sum_{ij} S_i^+ S_j^-$$

•For the same linear combination in the Hyperbolic family:

$$H_{Hyper} = \sum_i 2\varepsilon_i S_i^z + g\sum_{ij} \sqrt{\varepsilon_i \varepsilon_j} S_i^+ S_j^-$$

Application to Samarium isotopes

G.G. Dussel, S. Pittel, J. Dukelsky and P. Sarriguren, PRC 76, 011302 (2007)

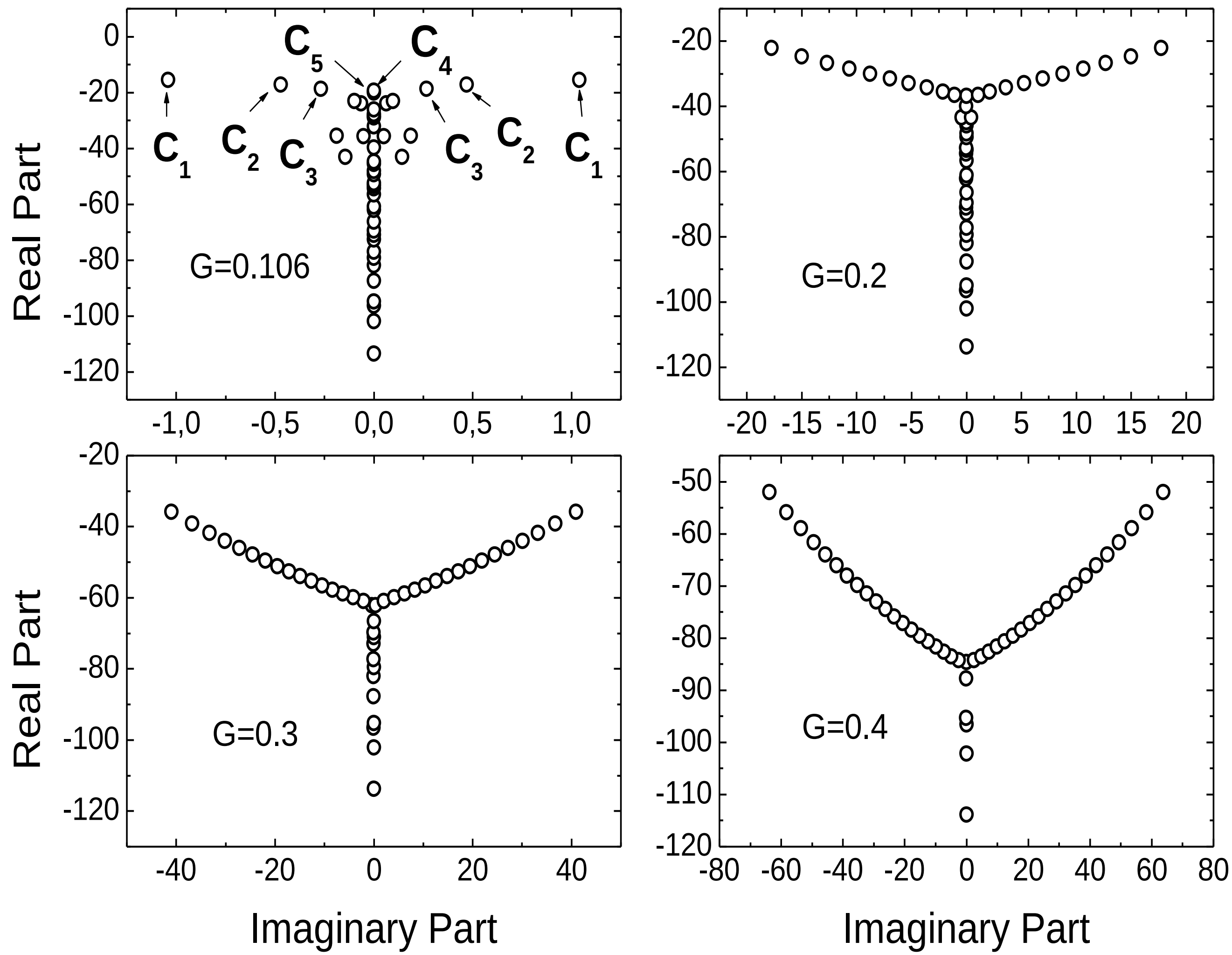
- $Z = 62$, $80 \leq N \leq 96$

- Selfconsistent Skyrme (SLy4) Hartree-Fock plus BCS in 11 harmonic oscillator shells. 40 to 48 pairs in 286 double degenerate levels. Dim. of the pairing Hamiltonian matrix ~ 10^{49} to 10^{53}.

- The strength of the pairing force is chosen to reproduce the experimental pairing gaps in ^{154}Sm (Δ_n=0.98 MeV, Δ_p= 0.94 MeV)

- g_n=0.106 MeV and g_p=0.117 MeV. A dependence $g=g_n/A$ is assumed for the isotope chain.

^{154}Sm



Real Part
Real Part
Imaginary Part
Imaginary Part
G=0.106
G=0.2
G=0.3
G=0.4
C5
C4
C1
C2
C3
C3
C2
C1

Correlations Energies

Mass	Ec(Exact)	Ec(PBCS	Ec(BCS+H)	Ec(BCS)
142	-4.146	-3.096	-1.214	-1.107
144	-2.960	-2.677	0.0	0.0
146	-4.340	-3.140	-1.444	-1.384
148	-4.221	-3.014	-1.165	-1.075
150	-3.761	-2.932	-0.471	-0.386
152	-3.922	-2.957	-0.750	-0.637
154	-3.678	-2.859	-0.479	-0.390
156	-3.716	-2.832	-0.605	-0.515
158	-3.832	-3.014	-1.181	-1.075

The Hyperbolic Model in Nuclear Structure

J. Dukelsky, S. Lerma H., L. M. Robledo, R. Rodriguez-Guzman, S. Rombouts, Phys. Rev. C 84, 061301(R) (2011)

The separable integrable Hyperbolic Hamiltonian

unphysical

$$H = \sum_i \eta_i S_i^z - G \sum_{i,j} \sqrt{\eta_i \eta_j} S_i^+ S_j^-$$

Redefining the 0 of energy $\eta_i = \varepsilon_i - \alpha$, absorbing the constant in the chemical potential μ

Exactly solvable H with non-constant matrix elements

$$H = \sum_i (\varepsilon_i - \mu) c_i^+ c_i - G \sum_{i,j} \sqrt{(\alpha - \varepsilon_i)(\alpha - \varepsilon_j)} c_i^+ c_{\bar{i}}^+ c_{\bar{j}} c_j$$

α is a new parameter that serves as an energy cutoff.

In BCS approximation:

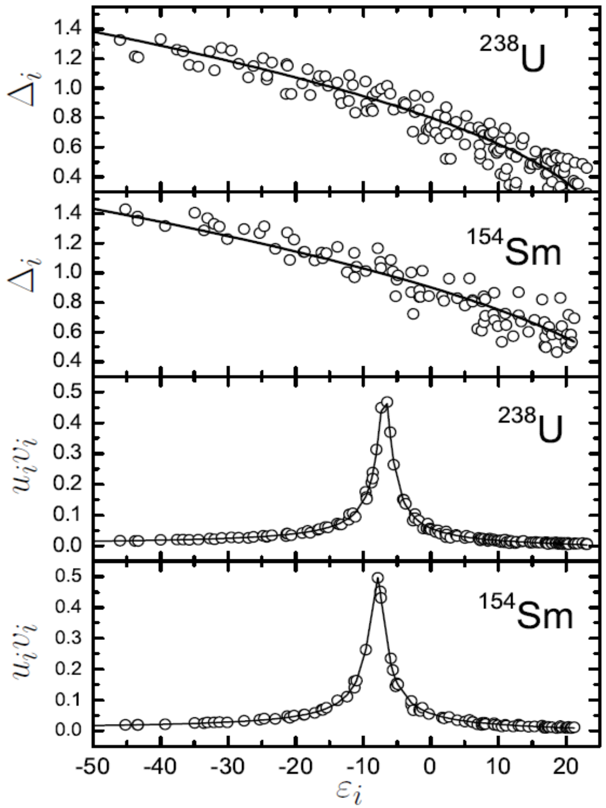
$$\Delta_i = G\sqrt{\alpha - \varepsilon_i} \sum_{i'} \sqrt{\alpha - \varepsilon_{i'}} u_{i'} v_{i'} = \Delta \sqrt{\alpha - \varepsilon_i}$$

The BCS Hamiltonian has

$$\Delta_i = \Delta$$

Mapping of the Gogny force in the Canonical Basis

We fit the pairing strength *G* and the interaction cutoff α to the paring tensor $u_i v_i$ and the pairing gaps Δ_i of the Gogny HFB eigenstate in the Hartree-Fock basis.



$$\Delta_i = G\sqrt{\alpha-\varepsilon_i}\sum_{i'}\sqrt{\alpha-\varepsilon_{i'}}\,u_{i'}v_{i'} = \Delta\sqrt{\alpha-\varepsilon_i}$$

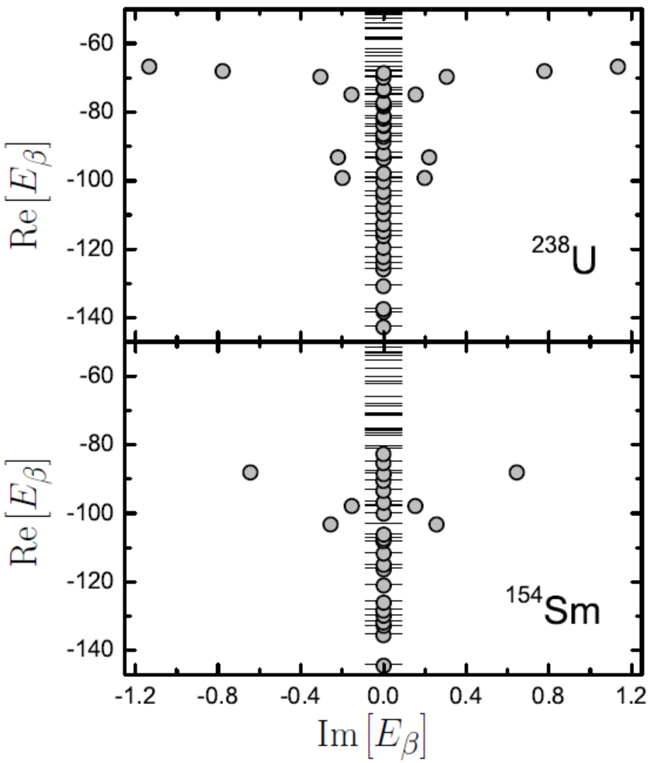
$$u_i v_i = \frac{\Delta\sqrt{\alpha-\varepsilon_i}}{2\sqrt{(\varepsilon_i-\mu)^2+(\alpha-\varepsilon_i)\Delta^2}}$$

Protons

○ Gogny

_ Hyperbolico

	M	L	D	G	α	Δ	E^{BCS}_{corr}	E^{Exa}_{corr}
^{154}Sm	31	95	9.9×10^{24}	2.2×10^{-3}	32.7	0.158	1.0164	2.9247
^{238}U	46	148	4.8×10^{38}	2.0×10^{-3}	25.3	0.159	0.503	2.651



Models derived from r = 1 RG [SU(2) and SU(1,1)]

- BCS or constant pairing Hamiltonian
- Generalized Pairing Hamiltonians (Fermion and Bosons)
- Central Spin Model (Quantum dot)
- Gaudin magnets (Quantum magnetism)
- Lipkin Model
- Two-level boson models (IBM, molecular, etc..)
- Atom-molecule Hamiltonians (Feshbach resonances in cold atoms)
- Generalized Jaynes-Cummings models.
- Breached superconductivity. LOFF and breached LOFF states.
- p-wave pairing in 2D lattices.
- Richardson-Gaudin-Kitaev model of topological supeconductivity.

Reviews: J.Dukelsky, S. Pittel and G. Sierra, Rev. Mod. Phys. 76, 643 (2004);
G. Ortiz, R. Somma, J. Dukelsky y S. Rombouts. Nucl. Phys. B 7070 (2005) 401

Exactly Solvable RG models for simple Lie algebras

Cartan classification of Lie algebras

rank	A_n ***su*(n+1)**	B_n ***so*(2n+1)**	C_n ***sp*(2n)**	D_n ***so*(2n)**
1	***su*(2), su(1,1) pairing**	*so*(3)~*su*(2)	*sp*(2) ~*su*(2)	*so*(2) ~*u*(1)
2	***su*(3) Three level Lipkins**	***so*(5), so(3,2) pn-pairing**	*sp*(4) ~*so*(5)	*so*(4) ~*su*(2)x*su*(2)
3	*su*(4) Wigner	*so(7) FDSM*	*sp*(6) FDSM	*so*(6)~*su*(4) color superconductivity
4	*su*(5)	*so(9)*	*sp*(8)	***so*(8) pairing T=0,1. Ginnocchio. S=3/2 fermions**

Exactly Solvable Pairing Hamiltonians

1) SU(2), Rank 1 algebra

$$H = \sum_i \varepsilon_i n_i - g \sum_{ij} P_i^+ P_j$$

2) SO(5), Rank 2 algebra

$$H = \sum_i \varepsilon_i n_i - g \sum_{ij\tau} P_{i\tau}^+ P_{j\tau}$$

J. Dukelsky, V. G. Gueorguiev, P. Van Isacker, S. Dimitrova, B. Errea y S. Lerma H. PRL 96 (2006) 072503.

3) SO(6), Rank 3 algebra

$$H = \sum_i \varepsilon_i n_i - g \sum_{ij\alpha} P_{i\alpha}^+ P_{j\alpha}$$

B. Errea, J. Dukelsky and G. Ortiz, PRA 79, 051603(R) (2009).

4) SO(8), Rank 4 algebra

$$H_{ST} = \sum_i \varepsilon_i n_i - g \sum_{ij\tau} S_{i\tau}^+ S_{j\tau} - g \sum_{ij\sigma} D_{i\sigma}^+ D_{j\sigma}$$

$$S_{i\tau}^+ = \frac{1}{\sqrt{2}} \left\{ a_i^+ a_i^+ \right\}_{0\tau}^{01}, \quad D_{i\sigma}^+ = \frac{1}{\sqrt{2}} \left\{ a_i^+ a_i^+ \right\}_{\tau 0}^{10}$$

S. Lerma H., B. Errea, J. Dukelsky and W. Satula. PRL 99, 032501 (2007).

Exact solution of the SO(8) model

$$E=\sum_{\alpha=1}^{M_1} e_\alpha+\sum_{i=1}^{L}\varepsilon_i u_i$$

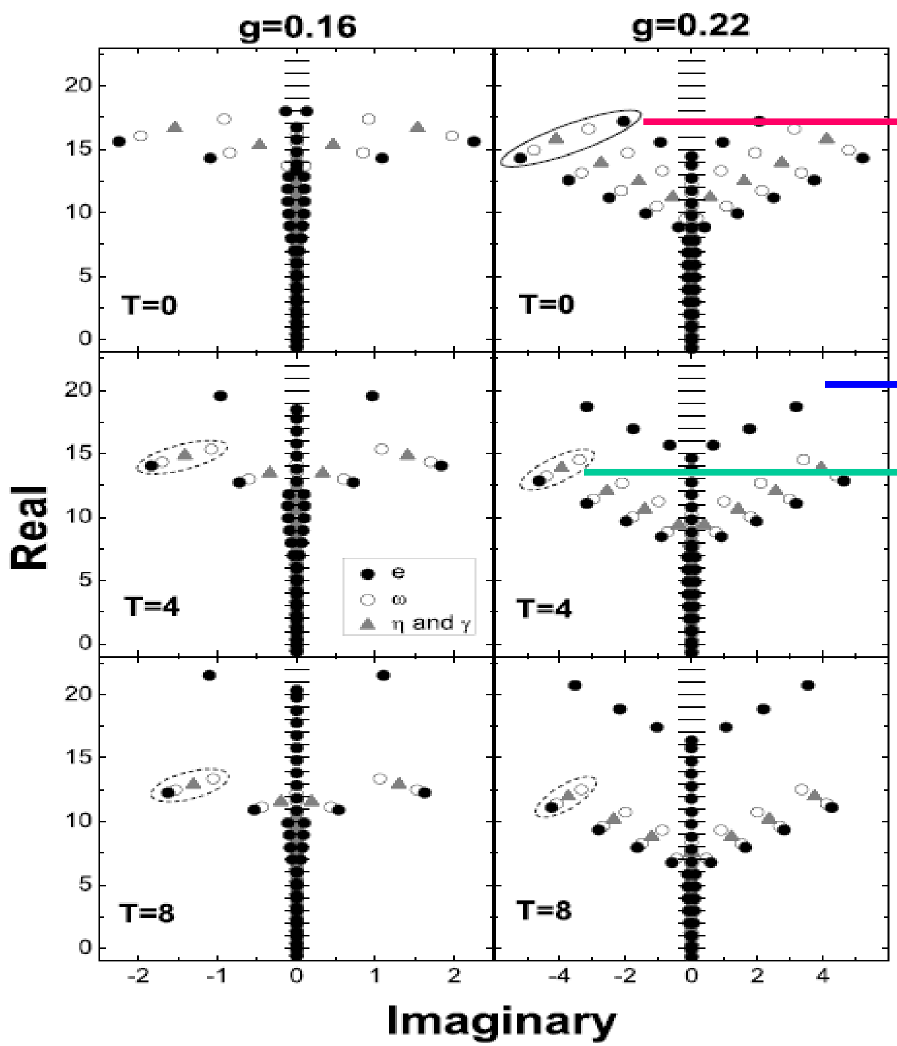
$$\sum_{\alpha(\neq\alpha')}^{M_1}\frac{2}{e_{\alpha'}-e_\alpha}-\sum_{\alpha}^{M_2}\frac{1}{\omega_{\alpha'}-e_\alpha}-\sum_{i}^{L}\frac{(2l_i+1)}{2\varepsilon_i-e_\alpha}+\frac{1}{g}=0$$

$$\sum_{\alpha'(\neq\alpha)}^{M_2}\frac{2}{\omega_{\alpha'}-\omega_\alpha}-\sum_{\alpha'}^{M_1}\frac{1}{e_{\alpha'}-\omega_\alpha}-\sum_{\alpha'}^{M_3}\frac{1}{\eta_{\alpha'}-\omega_\alpha}-\sum_{\alpha'}^{M_4}\frac{1}{\gamma_{\alpha'}-\omega_\alpha}+\sum_{\alpha'}^{M_1}\frac{1}{2\varepsilon_i-\omega_\alpha}=0$$

$$\sum_{\alpha'(\neq\alpha)}^{M_3}\frac{2}{\eta_{\alpha'}-\eta_\alpha}-\sum_{\alpha'}^{M_2}\frac{1}{\omega_{\alpha'}-\eta_\alpha}+\sum_{i}^{L}\frac{1}{2\varepsilon_i-\eta_\alpha}=0$$

$$\sum_{\alpha'(\neq\alpha)}^{M_4}\frac{2}{\gamma_{\alpha'}-\gamma_\alpha}-\sum_{\alpha'}^{M_2}\frac{1}{\omega_{\alpha'}-\gamma_\alpha}+\sum_{i}^{L}\frac{1}{2\varepsilon_i-\gamma_\alpha}=0$$

80 Nucleons in L=50 equidistant levels

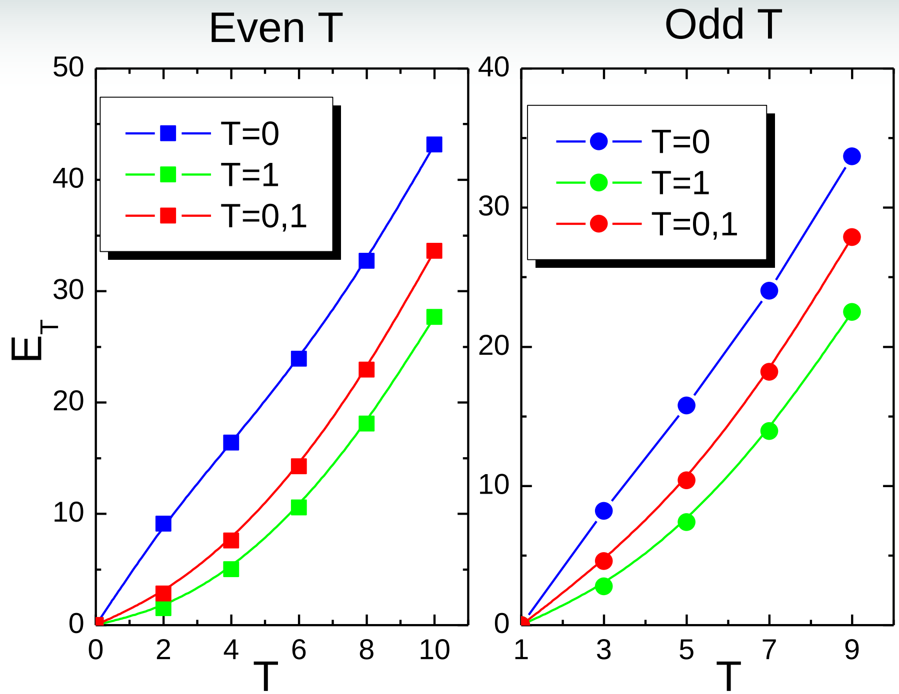


Quartet: 1*e*, 1ω, 1η, 1γ

n-n Cooper pair: 1*e*

p-p Cooper pair: 1*e*, 2ω, 1η, 1γ

G=0.22



Analysis of the nuclear symmetry energy vs T in terms of the Isocranking model (W. Satula and R. Wyss, PRL **86**, 4488 (2001) and **87**, 052504 (2001).

$$E_T^e = \frac{1}{2J_T} T\left(T+\lambda\right), \qquad E_T^o = \frac{1}{2J_T} T\left(T+\lambda\right) + \Delta E$$

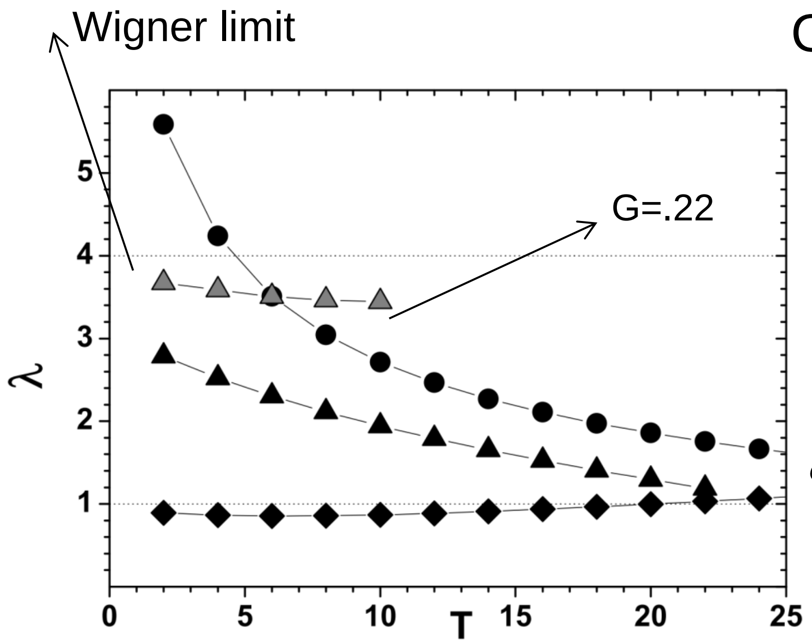
J_T: iso-MoI, λ: Linear enhancement factor (Wigner energy),

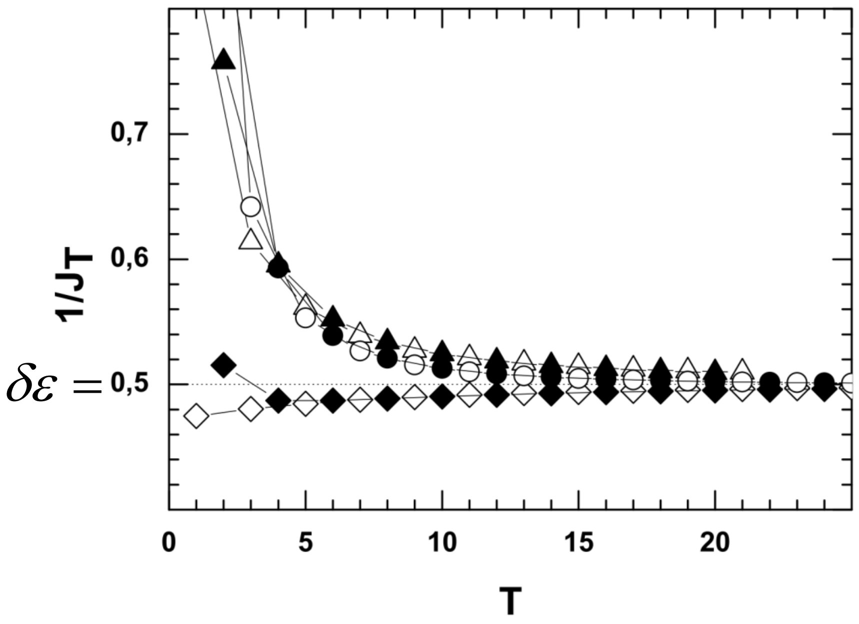
ΔE: 2qp excitation (ν=2)

Linear enhancement factor λ

Inverse of the Iso-MoI

G=0.16





T=0 circles, T=1diamonds, T=0,1 triangles. Solid (open) -> even (odd) T

Picket-Fence model and the thermodynamic limit of p-n BCS

G. F. Bertsch, J. Dukelsky, B. Errea, C. Esebbag, Ann. Phys. 325 (2019) 1340

Equidistant Ω single particle levels $\varepsilon_i = \dfrac{i}{2\Omega}, \, i = 1, \cdots, \Omega$

SU(4) symmetric pairing Hamiltonian

$$H_{ST} = \sum_i \varepsilon_i n_i - g \left[\sum_{ij\tau} S^+_{i\tau} S_{j\tau} + \sum_{ij\sigma} D^+_{i\sigma} D_{j\sigma} \right]$$

Quarter filling $\quad N = \Omega, \text{ with } g = 0.15 \rightarrow f \approx 0.54$

Thermodynamic limit $\quad N \rightarrow \infty, \, \Omega \rightarrow \infty, \quad \rho = \dfrac{N}{4\Omega} = \dfrac{1}{4}$

BCS equations:

$$4\int_0^{1/2} \left[1 - \frac{\varepsilon - \mu}{\sqrt{(\varepsilon - \mu)^2 + \Delta^2}} \right] d\varepsilon = 1, \quad \int_0^{1/2} \frac{\varepsilon - \mu}{\sqrt{(\varepsilon - \mu)^2 + \Delta^2}} \, d\varepsilon = \frac{1}{g}$$

Thermodynamic limit of the SO(8) exact solution

Unlike the SU(2) RG model, we cannot derive analytically the continuous limit. Proceed numerically by expanding the GS and quasiparticle energies as

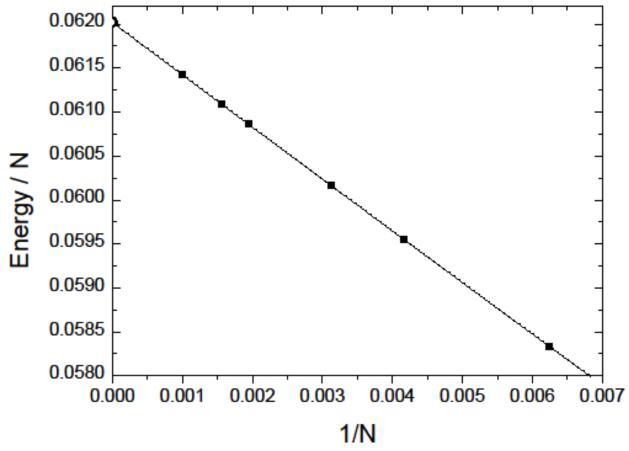
$$\frac{E_{GS}}{N} = a + \frac{b}{N} + \frac{c}{N^2} + \frac{d}{N^3} + \mathrm{O}\left(1/N^4\right)$$

$$E_q\left(4n\right) = E_{GS}\left(4n+1\right) - E_{GS}\left(4n\right)$$

$$\Delta_{o-e}\left(4n+1\right) = \frac{1}{2}\left[2E_{GS}\left(4n+1\right) - E_{GS}\left(4n\right) - E_{GS}\left(4n+2\right)\right]$$

$$\Delta_c = \frac{g}{8\Omega}\sum_{i=1,\sigma\tau}^{\Omega}\sqrt{n_{i\sigma\tau}\left(1-n_{i\sigma\tau}\right)}$$

$$\boxed{160 \le N \le 1000, \quad 40 \le n \le 250}$$



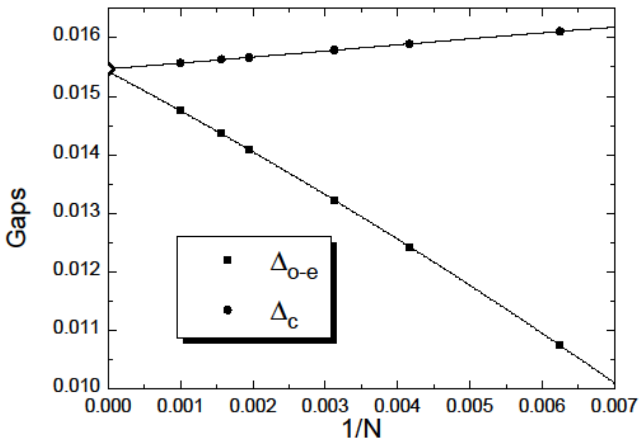
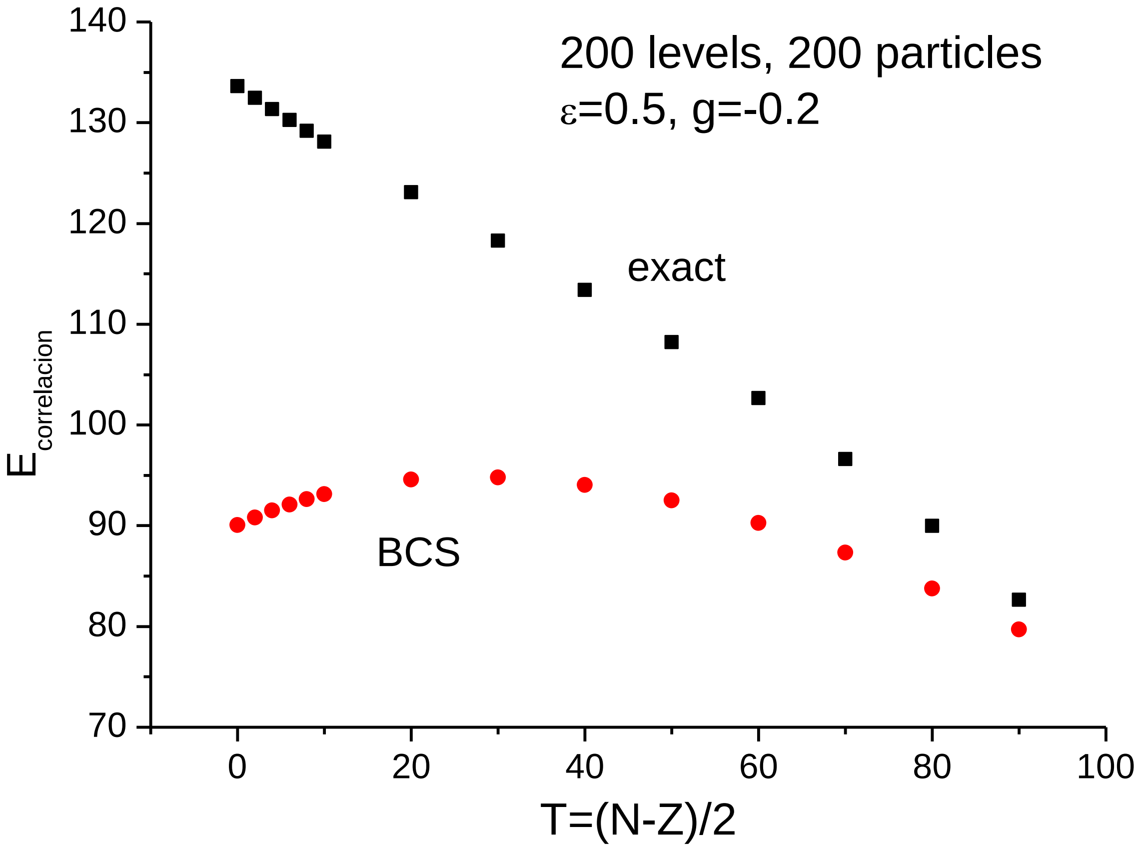


Table 1
Ground-state energy, quasiparticle energy, and different gaps as defined in the text for the $SU(4)$-symmetric pairing Hamiltonian in the $1/N$ cubic expansion, with a comparison to bulk BCS limit.

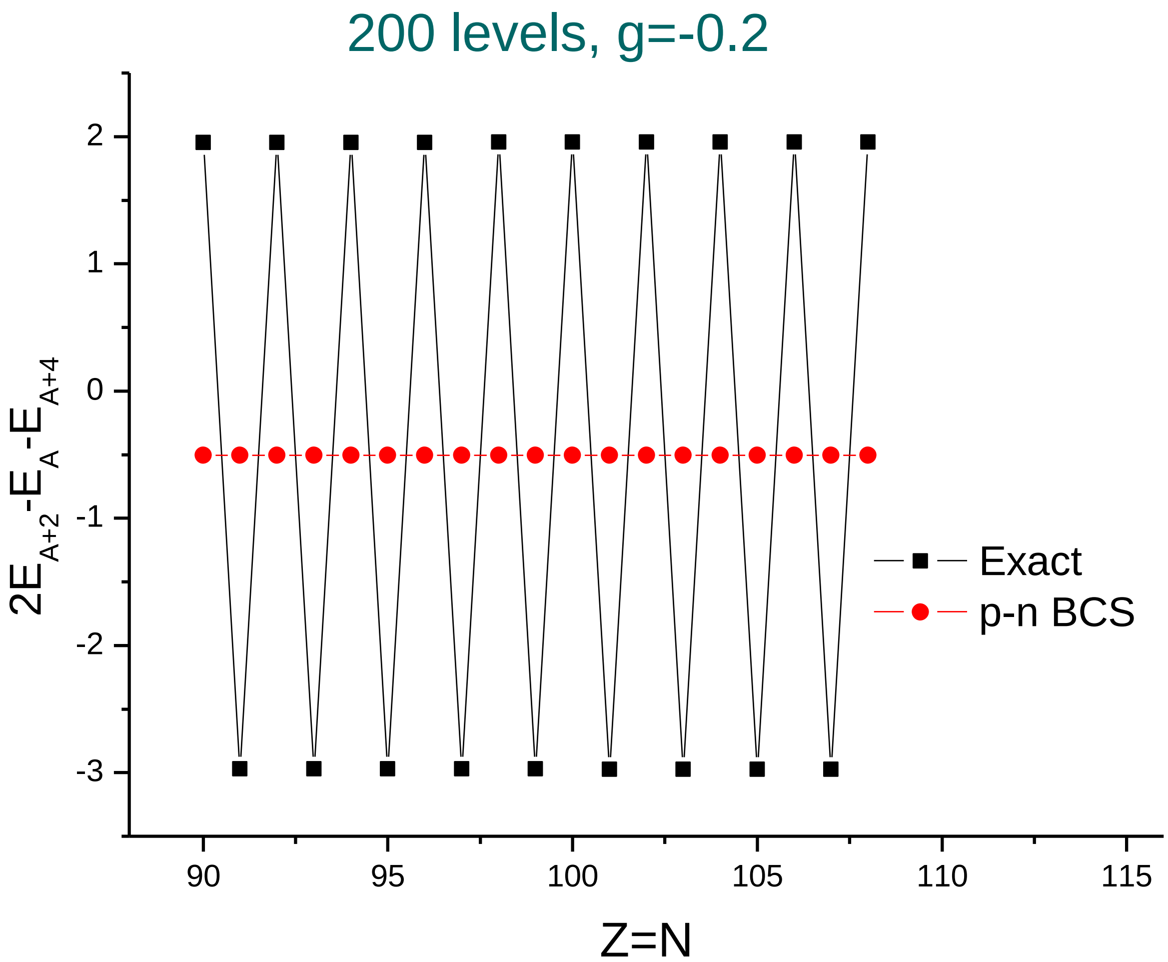
	Method	a	b	c	d
$\frac{E}{N}$	Exact	0.062022149	-0.597581	1.278831	-11.1571
	BCS	0.062022154			
E_q	Exact	0.140148	-0.479740	-10.0327	-1107.25
	BCS	0.140151			
Δ_{o-e}	Exact	0.0154637	-0.699890	-2.24642	-1066.63
	BCS	0.0154669			
Δ_c	Exact	0.0154672	0.0961964	2.59458	-257.910
	BCS	0.0154669			

T=0,1 Pairing


200 levels, 200 particles
ε=0.5, g=-0.2
exact
BCS
140
130
120
110
100
90
80
70
0
20
40
60
80
100
$E_{correlacion}$
T=(N-Z)/2

T=0,1 Pairing

Odd-Even Pair effect as a signal of quartet correlations



Summary

• For finite systems, PBCS improves significantly over BCS but it is still far from the exact solution. Typically, PBCS misses ~ 1 MeV in binding energy.

•The Isovector SO(5) and the SO(8) pairing models are excellent benchmark models to study different approximations dealing with quartet correlations, clusterization and condensation. The SO(8) model can also describe spin 3/2 cold atoms where nuclear physics could be explored in the lab.

•SO(5) has been used to test the QCM approximation in: N. Sandulescu, D. Negrea, J. Dukelsky, and C. W. Johnson Phys. Rev. C 85, 061303(R) (2012)

•The exact GS energy of the T=0,1 pairing Hamiltonian goes to p-n BCS energy in the thermodynamic limit. However, quartet correlations are important for finite systems.

•Alpha phases in nuclear matter require more realistic interactions: contact, schematic or realistic nuclear forces. Could they be explore with cold atoms in optical lattices?